Wiener Filter

$y[n]$ is process

$y[n] \to \boxed{g[n]} \to \underbrace{x[n]}$ observation.

We want to derive $\hat{h}[n]$ so that we can reconstruct the signal $y[n]$ from observations:

$x[n] \to \boxed{\hat{h}[n]} \to \hat{y}[n]$ ← estimate.

~~Min:~~ By minimizing errors:

$e[n] = \hat{y}[n] - y[n] \Rightarrow \min_{h[\cdot]} \varepsilon = E\{e^2[n]\}$

Noncausal DT Wiener Filter:

~~h here actually is the impulse response we are trying to estimate to get back y[n]~~

$\varepsilon = E\{e^2[n]\}$

$= E\left\{\left(\sum_{k=-\infty}^{\infty} h[k]x[n-k] - y[n]\right)^2\right\}.$

Want to minimize ε, so:

$\frac{\partial \varepsilon}{\partial h[m]} = 0$ for all values of m, where $h[m]$ is not restricted to be 0.

$\frac{\partial \varepsilon}{\partial h[m]} = E\left\{2\underbrace{\left(\sum_k h[k]x[n-k] - y[n]\right)}_{e[n]} x[n-m]\right\} = 0$

So, $E\{2e[n]x[n-m]\} = 0$, or

$R_{ex}[m] = 0 \Rightarrow e[n]$ and $x[n]$ are orthogonal.

cross-correlation.

For the optimal filter, the error should be orthogonal to all the data used to construct the estimate.

Also,

$R_{ex}[m] = E\{e[n]x[n-m]\}$

$= E\{(\hat{y}[n] - y[n])x[n-m]\}$

$= R_{\hat{y}x}[m] - R_{yx}[m] = 0.$

$\Rightarrow$ Optimal estimate would make it so

$R_{\hat{y}x}[m] = R_{yx}[m]$ for all appropriate m.

To derive the actual $h[n]$..

we have:

$\begin{cases} R_{\hat{y}x}[m] = h[m] * R_{xx}[m] \\ R_{yx}[m] = h[m] * R_{xx}[m] \end{cases}$

$\hookrightarrow$ ① $\sum_k h[k]R_{xx}[m-k] = R_{yx}[m] \Rightarrow$ This is a set of linear equations we can solve for.

Suppose $h[n]$ is restricted to $n \in [0, N-1]$, then:

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[-1] & \cdots & R_{xx}[1-N] \\ R_{xx}[1] & R_{xx}[0] & \cdots & R_{xx}[2-N] \\ \vdots & \vdots & \ddots & \vdots \\ R_{xx}[N-1] & R_{xx}[N-2] & \cdots & R_{xx}[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-1] \end{bmatrix} = \begin{bmatrix} R_{yx}[0] \\ R_{yx}[1] \\ \vdots \\ R_{yx}[N-1] \end{bmatrix}$$

This is the case when $h[n]$ is an FIR, for example.

When $h[n]$ is an IIR, we have infinite terms to solve!

~~So we~~ But that means eq. ① holds for $m \in (-\infty, \infty)$.

Use Z-transform:

$H(z)S_{xx}(z) = S_{yx}(z)$

$H(z) = \frac{S_{yx}(z)}{S_{xx}(z)}$. $\Rightarrow$ Z-transform of transfer function can be found

If the Input/Output process does not have a Z-transf, can use F.T domain calculations.

*Caveat: Going to frequency domain decouples the system.

$R_{ee}[m] = R_{yy}[m] - R_{y\hat{y}}[m] = R_{yy}[m] - h[m] * R_{xy}[m]$

The last step comes from:

$R_{y\hat{y}} = E\{\int y(t)\underbrace{[h(t-\tau)x(t-\tau)]}_{\hat{y}} d\tau\} = h[n] * R_{xy}[m]$

$= E\{$ ~~$\int y(t)h(t-\tau)$~~ $\}$

The MMSE $= \underline{R_{ee}[0]}$